1. (i) (5 marks) State the root test for the convergence of numerical series.

   (ii) (7 marks) Determine whether the series $\sum_{n=1}^{\infty} n^3 e^{-\sqrt{n}}$ converges. Justify your answer.

   (iii) (8 marks) Determine whether the series

$$\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{7}} \cdots$$

   converges. The signs occur in blocks increasing in length by one at each step. Justify your answer.

2. Let $(n_j)_{j=1}^{\infty}$ be a sequence of positive integers such that (a) $n_j < n_{j+1}$ for $j = 1, 2, 3, \ldots$, (b) $n_j | n_{j+1}$, that is, n_j divides n_{j+1} in $\mathbb{Z}$ for $j = 1, 2, 3, \ldots$ and (c) for every integer $q \in \mathbb{N}$, there exists $j \in \mathbb{N}$ such that $q | n_j$.

   (i) (8 marks) Show from first principles that the series $\sum_{j=1}^{\infty} (-1)^{j-1} \frac{1}{n_j}$ converges.

   (ii) (8 marks) Show that the sum of the series in (i) is an irrational number.

   (iii) (4 marks) Deduce that $\cos(\sqrt{2})$ is irrational.

3. (i) (3 marks) Define the upper and lower Riemann sums.

   (ii) (3 marks) State Riemann's condition for integrability.

   (iii) (7 marks) Let f be a continuous function on $[0, 1]$. Show that $\int_0^1 f(x)\ dx$ exists and that $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \int_0^1 f(x)\ dx$.

   (iv) (7 marks) By writing $\ln\left(\frac{(2n)!}{n^n(n!)}\right) = \sum_{k=1}^{n} \ln\left(1 + \frac{k}{n}\right)$ determine the limit

$$\lim_{n\to\infty} \left\{\frac{(2n)!}{n^n(n!)}\right\}^{\frac{1}{n}}$$

4. (i) (4 marks) Define the concept of uniform convergence of a sequence of functions.
   (ii) (4 marks) State, but do not prove, the Cauchy Criterion for uniform convergence.

   Let $\sum_{k=0}^{\infty} a_k$ be a convergent series and denote $t_n = \sum_{k=n}^{\infty} a_k$.

   (iii) (4 marks) Establish the summation by parts formula

$$\sum_{k=p}^{q} a_k x^k = t_p x^p - t_{q+1} x^q - (1-x) \sum_{k=p+1}^{q} t_k x^{k-1}.$$

   (iv) (4 marks) Establish the estimate $\sup_{0 \le x \le 1} \left| \sum_{k=p}^{q} a_k x^k \right| \le 3 \sup_{k \ge p} |t_k|$.

   (v) (4 marks) Deduce that $\sum_{k=0}^{\infty} a_k x^k$ converges uniformly for $0 \le x \le 1$.

5. (i) (4 marks) State (but do not prove) the Fundamental Theorem of Calculus.

   Let φ be a function $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ with continuous second derivative.

   (ii) (8 marks) Show that

$$\varphi(x) = \varphi(a) + \varphi'(a)(x-a) + \int_{\xi=a}^{x} (x-\xi)\varphi''(\xi)\, d\xi$$

   for all $a, x \in \mathbb{R}$.
   (iii) (8 marks) Show that

$$\varphi(t) = t\varphi(1) + (1-t)\varphi(0) - \int_{\xi=0}^{1} \theta(t,\xi)\varphi''(\xi)\, d\xi$$

   for all $t \in [0,1]$ and where

$$\theta(t,\xi) = \begin{cases} t(1-\xi) & \text{if } 0 \le t \le \xi \le 1, \\ \xi(1-t) & \text{if } 0 \le \xi \le t \le 1. \end{cases}$$

6. (i) (4 marks) Let $a_{m,n} \geq 0$ for $m, n \in \mathbb{N}$. State (but do not prove) a theorem giving conditions for $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{m,n} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{m,n}$.

(ii) (4 marks) Let $a_{m,n} \in \mathbb{R}$ for $m, n \in \mathbb{N}$. State (but do not prove) a theorem giving conditions for $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{m,n} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{m,n}$.

Let $c_{m,n} = \begin{cases} \dfrac{1}{m^2 - n^2} & \text{if } m \neq n, \\ 0 & \text{if } m = n. \end{cases}$

(iii) (6 marks) Use the identity $\dfrac{1}{m^2 - n^2} = \dfrac{1}{2m}\left(\dfrac{1}{m-n} + \dfrac{1}{m+n}\right)$ to establish that

$$\sum_{n=1}^{r} c_{m,n} = -\frac{3}{4m^2} + \frac{1}{2m} \sum_{k=r-m+1}^{r+m} \frac{1}{k}$$

for $r \geq 2m$.

(iv) (6 marks) Deduce that $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{m,n} \neq \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_{m,n}$ and explain why the conditions you have given in (ii) are not satisfied.

7. (i) (2 marks) State (but do not prove) a theorem about differentiation under the integral sign.

(ii) (2 marks) State (but do not prove) a theorem about the interchange of limit and integral.

Let $f(t) = \displaystyle\int_{x=\frac{\pi}{3}}^{\frac{2\pi}{3}} (\sin x)^t \, dx$.

(iii) (4 marks) Obtain an integral formula for $f^{(k)}(0)$, for $k = 0, 1, 2, \ldots$

(iv) (4 marks) Use the formula you have found in (iii) to show $|f^{(k)}(0)| \leq \dfrac{\pi}{3}\left|\ln \dfrac{2}{\sqrt{3}}\right|^k$.

(v) (4 marks) Find the radius of convergence ρ of the power series $\displaystyle\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} t^k$.

(vi) (4 marks) Show that $f(t) = \displaystyle\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} t^k$ for all t with $|t| < \rho$.

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FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 189-255B

Analysis II

Examiner: Professor S. W. Drury
Associate Examiner: Professor J. R. Choksi

Date: Wednesday, 28 April 1999
Time: 2: 00 pm. – 5: 00 pm.

INSTRUCTIONS

All seven questions should be attempted for full credit.

This is a closed book examination.
Write your answers in the booklets provided.
No calculators are allowed.

All questions are of equal weight; each is worth 20 marks.
The exam will be marked out of a total of 140 marks
and subsequently scaled to a percentage.

This exam comprises the cover and 3 pages of questions.