1. Define:

(a) (i) the sequence of functions (f_n) converges uniformly on S to a function $f : S \to \mathbf{R}$;

(ii) the infinite series $\sum_{n=1}^{\infty} f_n$ converges uniformly on S.

(b) Let f be a bounded function defined on $[a, b]$, $(-\infty < a < b < \infty)$.
Define:

(i) Upper (Darboux) sum $U(P)$ of f with respect to the partition P of $[a, b]$;

(ii) Upper and lower (Darboux) integrals $\overline{\int_a^b} f dx$ and $\underline{\int_a^b} f dx$ respectively;

(iii) f is integrable on $[a, b]$.

2. (a) Prove the comparison test: Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series of non negative terms. If $a_n \leq b_n$ for $n \geq N$ and $\sum_{n=1}^{\infty} b_n$ converges, so does $\sum_{n=1}^{\infty} a_n$; if the series $\sum_{n=1}^{\infty} a_n$ is divergent, so does $\sum_{n=1}^{\infty} b_n$.

(b) Show that $\sum_{n=1}^{\infty} \frac{1}{n} \log\left(1 + \frac{1}{n}\right)$ is convergent.

3. (a) Let $\sum_{n=1}^{\infty} a_n$ be a convergent series. If $0 < b_{n+1} \leq b_n$ for $n \in \mathbb{N}$, prove that $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

(b) Suppose that $\sum_{n=1}^{\infty} \frac{a_n}{n^p}$ is convergent. Show that $\sum_{n=1}^{\infty} \frac{a_n}{n^q}$ is convergent if $q > p$.

4. Let $A \subset \mathbb{R}$, suppose that $f_n : A \to \mathbb{R}$, and $|f_n(x)| \leq M_n$ for $x \in A$, $n \in \mathbb{N}$. If $\lim_{n \to \infty} f_n = f$ uniformly on A, prove that:

(a) (i) f is bounded on A;
(ii) $|f_n(x)| \leq M$ for all $n \in \mathbb{N}$ and $x \in A$.

(b) If $g : \mathbb{R} \to \mathbb{R}$ is continuous, the sequence of the composite functions $(g \circ f_n)(x) = g(f_n(x))$, $n \in \mathbb{N}$, converges uniformly to $g \circ f$ on A.

5. (a) Suppose that $f_0 : [0,a] \to \mathbb{R}$ is continuous. If $f_n(x) := \int_0^x f_{n-1}(t)dt,\ 0 \le x \le a$, prove that $(f_n),\ n \in \mathbb{N}$, converges uniformly to the zero function on $[0,a]$.

   (b) If $f(x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n}}\ \text{Arctan}\frac{x}{\sqrt{n}}$, show that $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n+x^2} = f'(x)$. (Theorems used in your argument should be fully stated.)

6. (a) State and prove Abel's limit theorem.

   (b) Justify the formula

$$\int_0^1 \frac{t^{p-1}}{1+t^q}dt = \sum_{k=0}^{\infty}(-1)^k\frac{1}{p+kq},$$

   where p and q are positive integers.

7. (a) State a condition equivalent to the Riemann integrability of a bounded function defined on a closed and bounded interval $[a,b],\ (a < b)$.

   (b) Prove that every continuous function defined on $[a,b]$ is Riemann integrable.

8. (a) Let f be continuous on $[0,1]$. If $g_n(x) = f(x^n)$ for $n \in \mathbb{N}$, show that

$$\lim_{n\to\infty}\int_0^1 g_n(x)dx = f(0).$$

   (b) Let f be Riemann integrable and g be continuous on $[a,b],\ (a < b)$. If $g' = f$ for $x \in (a,b)$ prove that $\int_a^b f dx = g(b) - g(a)$.

FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 189-255B

ANALYSIS II

Examiner: Professor R. Vermes
Associate Examiner: Professor J.R. Choksi

Date: Wednesday, April 30, 1997
Time: 2:00 P.M. - 5:00 P.M.

This exam comprises the cover and 2 pages of questions.