FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS MATH 255

Honours Analysis 2

Examiner: Professor S. W. Drury — Date: Wednesday, 22 April 2009

Associate Examiner: Professor V. Jaksic — Time: 2: 00 pm. – 5: 00 pm.

INSTRUCTIONS

Answer all questions in the booklets provided.
This is a closed book examination.
Calculators are not permitted.
Both regular and translation dictionaries are permitted.

Each question is worth 20 points. The entire exam is worth 120 points.

This exam has 6 questions and 3 pages

1. (i) (4 points) Define the term *metric space*.
(ii) (4 points) Define the term *open subset* of a metric space.
(iii) (4 points) Define the term *closed subset* of a metric space.
(iv) (8 points) Show from first principles that a subset of a metric space is open if and only if its complement is closed.

2. Let $\sum_{n=1}^{\infty} a_n$ be a convergent series of *real* numbers.
(i) (9 points) Which of the following series *necessarily* converges (no justification required)

$$\text{(a)} \sum_{n=1}^{\infty} a_n^3, \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{a_n}{\sqrt{n}}, \qquad \text{(c)} \sum_{n=1}^{\infty} (-1)^n \frac{a_n^2}{\sqrt{n}}.$$

(ii) (11 points) For just *one* of (a), (b) or (c) above, give a detailled justification of your assertion. If the series necessarily converges then give a proof. If it need not converge, then give an explicit example where it does not, justifying both the convergence of $\sum_{n=1}^{\infty} a_n$ and the failure to converge of your chosen series.

3. (i) (5 points) Define the term *Riemann partition*.
(ii) (5 points) Define the upper and lower Riemann sums $U(P, f)$ and $L(P, f)$ for a Riemann partition P.

(iii) (10 points) Let $f : [0, 1] \longrightarrow \mathbb{R}$ be given by $f(x) = \begin{cases} 1 & \text{if } x = 0, \\ x \left\lceil \dfrac{1}{x} \right\rceil & \text{if } 0 < x \leq 1. \end{cases}$ Prove that f is Riemann integrable on $[0, 1]$.

Note: For a real number t, the notation $\lceil t \rceil$ denotes the unique *integer* k such that $k-1 < t \leq k$.

4. For each of the following sequences of functions defined on $[0, \infty[$ determine the pointwise limit. Determine also whether convergence is uniform on $[0, \infty[$. Justify your answer.

(i) (6 points) $f_n(x) = \dfrac{\lceil n^2 x \rceil}{n^2 + x^2}$.

Note: For a real number t, the notation $\lceil t \rceil$ denotes the unique *integer* k such that $k-1 < t \leq k$.

(ii) (7 points) $f_n(x) = \left(1 + \dfrac{x}{n}\right)^{-n}$.

(iii) (7 points) $f_n(x) = ne^{-nx}\Big(\sin(x)\Big)^2$.

5. (i) (5 points) State a theorem about composition of power series. What assertions can be made about the radius of convergence of the composed series?
(ii) (5 points) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $(f(x))^3 = \cos(x)$. Show that f has a power series expansion about $x = 0$ with strictly positive radius.
(iii) (5 points) Find explicitly the expansion in (ii) as far as the term in x^4.
(iv) (5 points) Show that the radius of convergence of the expansion in (ii) cannot exceed $\dfrac{\pi}{2}$.

6. (i) (3 points) State (but do not prove) a theorem about differentiation under the integral sign.
(ii) (3 points) State (but do not prove) a theorem about the interchange of limit and integral.
(iii) (3 points) State (but do not prove) the Fundamental Theorem of Calculus.

Observe that $\frac{\partial}{\partial t}\Big(e^{\frac{1}{2}(t^2-x^2)}\cos(tx)\Big) = \frac{\partial}{\partial x}\Big(e^{\frac{1}{2}(t^2-x^2)}\sin(tx)\Big)$.

(iv) (4 points) Prove that $\frac{\partial}{\partial t}\int_0^a e^{\frac{1}{2}(t^2-x^2)}\cos(tx)dx = e^{\frac{1}{2}(t^2-a^2)}\sin(ta)$.

(v) (3 points) Deduce that $\int_0^a e^{\frac{1}{2}(t^2-x^2)}\cos(tx)dx - \int_0^a e^{-\frac{1}{2}x^2}dx = \int_0^t e^{\frac{1}{2}(s^2-a^2)}\sin(sa)ds$.

(vi) (4 points) Further deduce that $\int_0^\infty e^{\frac{1}{2}(t^2-x^2)}\cos(tx)dx = \int_0^\infty e^{-\frac{1}{2}x^2}dx$ for each fixed $t \in \mathbb{R}$.

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