1. (i) (10 marks) State and prove the Cauchy–Schwarz inequality

   (ii) (10 marks) Let $a_1, a_2, \ldots, a_n$ be positive numbers. By writing $a_1 = (a_1 a_2^{-\frac{1}{2}})a_2^{\frac{1}{2}}$, $a_2 = (a_2 a_3^{-\frac{1}{2}})a_3^{\frac{1}{2}}$, …, $a_n = (a_n a_1^{-\frac{1}{2}})a_1^{\frac{1}{2}}$ or otherwise, show that

$$a_1 + a_2 + \cdots + a_{n-1} + a_n \le a_1^2 a_2^{-1} + a_2^2 a_3^{-1} + \cdots + a_{n-1}^2 a_n^{-1} + a_n^2 a_1^{-1}$$

2. (i) (6 marks) Describe Riemann's Criterion for Integrability.

   (ii) (7 marks) If f is a Riemann Integrable function on $[0, 1]$ show that the function $|f|$ defined by $|f|(x) = |f(x)|$ is also Riemann Integrable on $[0, 1]$.

   (iii) (7 marks) Let

$$g(x) = \begin{cases} 0 & \text{if } x \text{ is irrational,} \\ \dfrac{1}{q} & \text{if } x = \dfrac{p}{q} \text{ in lowest terms with } p \text{ and } q \text{ integers.} \end{cases}$$

   Is g Riemann Integrable on $[0, 1]$? Justify your answer.

3. (i) (5 marks) Let $f(x) = e^{x^2} \displaystyle\int_0^x e^{-t^2}\,dt$. How is it possible to assert on theoretical grounds that f has a power series expansion about $x = 0$ with infinite radius?

   (ii) (5 marks) Show that $f'(x) = 1 + 2xf(x)$.

   (iii) (5 marks) Find the power series expansion of f about $x = 0$ as far as the term in x^7.

   (iv) (5 marks) Use the ratio test to verify that the radius of the series you have found is indeed infinite.

4. For each of the following sequences of functions defined on $\mathbb{R}$ determine (a) if a pointwise limit exists everywhere on $\mathbb{R}$, (b) if a uniform limit exists on each bounded subset of $\mathbb{R}$ and (c) if a uniform limit exists on $\mathbb{R}$.

   (i) (7 marks) $f_n(x) = \left(1 + \dfrac{x}{n}\right)^n$.

   (ii) (6 marks) $f_n(x) = \dfrac{x}{1 + nx^2}$.

   (iii) (7 marks) $f_n(x) = \cos(nx^2)$.

   Justify your answers.

5. Let $a_n > 0$ and $\sum_{n=1}^{\infty} a_n < \infty$. For each of the following statements, either provide a proof that the statement necessarily holds, or an example of a specific instance where it does not.

(i) (7 marks) $\sum_{n=1}^{\infty} n^2 a_n^3 < \infty$.

(ii) (7 marks) $\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \le 1$.

(iii) (6 marks) $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n^2} < \infty$.

6. (i) (6 marks) State the Fundamental Theorem of Calculus.

(ii) (7 marks) Let g and h be two differentiable functions such that

- $g(0) = h(0)$
- $g'(x) \le h'(x)$ for $x > 0$

Show that $g(x) \le h(x)$ for $x \ge 0$.

(iii) (7 marks) Suppose that f is a differentiable function such that $f(0) = 0$ and $0 < f'(x) \le 1$ for all $x > 0$. Show that for $x \ge 0$

$$\int_0^x \Big(f(t)\Big)^3 dt \le \Big(\int_0^x f(t)dt\Big)^2.$$

Hint: Apply (ii) twice (at least).

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FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS MATH255

Analysis2

Examiner: Professor S. W. Drury
Associate Examiner: Professor K. N. GowriSankaran

Date: Friday, April 23, 2004
Time: 2: 00 pm. – 5: 00 pm.

INSTRUCTIONS

All six questions should be attempted for full credit.

This is a closed book examination.
Write your answers in the booklets provided.
No calculators are allowed.

All questions are of equal weight; each is worth 20 marks.
The exam will be marked out of a total of 120 marks
and subsequently scaled to a percentage.

This exam comprises the cover and 2 pages of questions.