1. Let A be a $(n \times n)$-matrix over the field F.
   (a) Define the <u>nullspace</u> N_A of A and the <u>column space</u> C_A of A.
   (b) Show that if $A^2 = 0$, then $C_A \subset N_A$.
   (c) Show that if $A^2 = A$, then $C_A \cap N_A = 0$ and $F^n = C_A + N_A$.

2. Let

$$A = \begin{pmatrix} 1 & 1 & 0 & 2 \\ -1 & 1 & -2 & 0 \\ -1 & 0 & -1 & -1 \\ 0 & -1 & 1 & -1 \end{pmatrix} .$$

   With N_A and C_A as in question 1, find $\dim N_A$ and $\dim C_A$. Determine a basis for $C_A \cap N_A$.

3. Let V be the vectorspace of complex polynomials of degree ≤ 2. For $z \in \mathbb{C}$, define

$$\varepsilon_z : V \to \mathbb{C}$$

   by $\varepsilon_z(p) = p(z)$.
   (a) Show that ε_z is a linear map.
   (b) Let $z_1 = 1,\ z_2 = -1$ and $z_3 = i$. Let $\lambda_i = \varepsilon_{z_i}$. Show that $\{\lambda_1, \lambda_2, \lambda_3\}$ is a basis of the vectorspace $\hat{V}$ of linear maps from V to $\mathbb{C}$.
   (c) Express ε_0 as a linear combination of $\lambda_1, \lambda_2, \lambda_3$.

4. Let A be the $(n \times n)$-matrix

$$A = \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & 1 & 0 \end{pmatrix} .$$

   (A has entries 1 just above and below the diagonal, i.e. $a_{ij} = 1$ if $|i - j| = 1$, and all other entries are zero.) Compute $\det(A)$ (in terms of n).

5. Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

   (a) Determine the characteristic polynomial χ_A, the eigenvalues of A and bases for the eigenspaces.

   (b) Find the minimal polynomial μ_A and determine the Jordan canonical form of A. Justify your answer.

   (c) Compute A^{10}. Hint: $A^r = ((A - I) + I)^r$.

6. Let V be $\mathbb{R}^4$ with the standard inner product. Let U be the subspace generated by $(1, 1, 1, 1)$, and $(0, 1, 1, 1)$.

   (a) Find orthonormal bases for U and for $U^\perp$.

   (b) Find the vector $u \in U$ that is closest to $e_1 = (1, 0, 0, 0)$.

7. Find an orthogonal matrix P that diagonalizes the quadratic form

$$Q(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3 .$$

8. Let

$$A = \begin{pmatrix} 0 & 0 & 3+4i \\ 0 & 0 & 0 \\ 3-4i & 0 & 0 \end{pmatrix} .$$

   (a) Compute A^*. Is A Hermitian? Is A normal? Is A unitary? Justify your answer.

   (b) Find a unitary matrix U so that U^*AU is diagonal. Give an a priori reason why this is possible.

FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 189-247B

LINEAR ALGEBRA

Examiner: Professor K.P. Russell	Date: Thursday, April 24, 1997
Associate Examiner: Professor J.R. Choksi	Time: 2:00 P.M. - 5:00 P.M.

This exam comprises the cover and 2 pages of questions.