1. Let $f : [a, b] \to \mathbf{R}$ be a bounded function, and let P and Q be partitions of $[a, b]$.

   (a) Define the property: Q is a refinement of P.

   (b) If Q is a refinement of P, state the inequalities which relate all four of the quantities $U(P; f)$, $U(Q; f)$, $L(P; f)$, and $L(Q; f)$.

   (c) Suppose that there exist two partitions P_1 and P_2 of $[a, b]$ with

   $$U(P_1; f) - L(P_2; f) \leq 1/10.$$

   Prove that there exists a partition P_3 of $[a, b]$ with

   $$U(P_3; f) - L(P_3; f) \leq 1/10.$$

   (Do not assume that f is integrable.)

2. (a) Simplify the sum

   $$\sum_{k=1}^{n} (y_k - y_{k-1})$$

   and prove your result by induction on n. (Here $n \in \mathbf{N}$ and $y_0, y_1, \ldots, y_n$ are any real numbers).

   (b) Assuming the result of (a), prove the identity

   $$\sum_{k=0}^{n-1} x^k \quad = \quad \frac{x^n - 1}{x - 1}, \qquad x \neq 1.$$

   Hint: multiply both sides by $x - 1$.

   (c) Prove that the series

   $$\sum_{k=0}^{\infty} x^k$$

   converges for any $x \in (-1, 1)$ and find its sum.

3. Let $F : [a, b] \to \mathbf{R}$ have a continuous derivative $F' : [a, b] \to \mathbf{R}$ and let $P = (x_0, x_1, \ldots, x_n)$ be a partition of $[a, b]$.

   (a) Prove that for each $k = 1, 2, \ldots, n$

   $$m'_k(x_k - x_{k-1}) \quad \leq \quad F(x_k) - F(x_{k-1}) \quad \leq \quad M'_k(x_k - x_{k-1}),$$

   where m'_k and M'_k are the min and max of F' on $[x_{k-1}, x_k]$.

   (b) Assuming the results of 2(a) and 3(a), prove that

   $$L(P; F') \quad \leq \quad F(b) - F(a) \quad \leq \quad U(P; F').$$

4. State examples of the following and briefly explain each example.

   (a) A bounded function on [0,1] which is not Riemann integrable.

   (b) A function which is Riemann integrable on [0,1] but not continuous on [0,1].

   (c) A Riemann integrable function $f : [0,2] \to \mathbf{R}$ such that the indefinite integral of f is not differentiable at $x = 1$.

5. For $n = 1, 2, \ldots$ let f_n be the functions defined by

$$f_n(x) \quad = \quad \frac{xne^{nx}}{1 + ne^{nx}}, \qquad 1 \le x \le 2.$$

   (a) For $x \in [1,2]$, let $f(x) = \lim_{n\to\infty} f_n(x)$. Evaluate $f(x)$ and determine whether or not the limit is uniform on $[1,2]$ (prove your assertion).

   (b) Evaluate

$$\lim_{n\to\infty} \int_1^2 f_n(x)dx,$$

   and justify your work.

6. In this problem you may assume the result that $\int_0^\pi \sin^2(nx)dx = \pi/2$ for any $n \in \mathbf{N}$.

   (a) Prove that the series

$$\sum_{n=1}^{\infty} \frac{\sin^2(nx)}{n^2}$$

   converges uniformly on $\mathbf{R}$.

   (b) Let $f(x)$ denote the sum of the series in (a). Prove that f is Riemann integrable on $[0,\pi]$ and that

$$\int_0^\pi f(x)dx \quad = \quad \frac{\pi}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

7. For each statement below, state whether it is True or False. If it is False, give a counterexample. For the first 3 statements, suppose that the power series $\sum_{k=0}^{\infty} c_k x^k$ has radius of convergence R and that $0 < R < \infty$. Then this power series

   (a) converges absolutely for each $x \in (-R, R)$.

   (b) converges uniformly on $(-R, R)$.

   (c) defines a function $p(x)$ which is differentiable at each point $x \in (-R, R)$.

   (d) If a sequence of functions f_n converges uniformly to a function f on $[0,1]$ and if each f_n is differentiable on $[0,1]$, then f is differentiable on $[0,1]$.

FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 189-243B

REAL ANALYSIS

Examiner: Professor I. Klemes
Associate Examiner: Professor S. Drury

Date: Wednesday, April 21, 1999
Time: 2:00 pm - 5:00 pm

INSTRUCTIONS

NO CALCULATORS PERMITTED
Show your work.
Answer all 7 questions.
Keep this exam paper.

This exam comprises the cover and 2 pages of questions.