1. (a) Define $f : I \to \mathbb{R}$ is a continuous function where I is an interval.

   (b) Suppose $f : [a, b] \to \mathbb{R}$ is continuous and is such that $f(x)$ is never rational for any $x \in [a, b]$. If $f(a) = \sqrt{2}$, find $f(b)$. Justify.

   (c) If f and g are both continuous defined on the same interval $I \subseteq \mathbb{R}$.
   Let $A = \{x : f(x) = g(x)\}$. If $x_n \in A$ and $x_n \to t \in I$, show that $t \in A$.

2. (a) Let A and B two sets that are bounded above. Prove that $C = \{x+y : x \in A, y \in B\}$ is also bounded above. Further show that

$$\sup C = \sup A + \sup B.$$

   (b) Let $S = \{x : x = \left(-\frac{1}{2}\right)^m - \frac{3}{n},\ n, m \in N\}$. Find the sup and inf of S.

3. Consider the polynomial $p(x) = x^3 - 7x + 4$. Construct a suitable contractive sequence and deduce that there is a solution of $p(x) = 0$ in $(0, 1)$.

4. (a) Define "$f : A \to \mathbb{R}$ is uniformly continuous".

   (b) Let $A \subset \mathbb{R}$, $f : A \to \mathbb{R}$ be uniformly continuous. If (x_n) is a Cauchy sequence with $x_n \in A$, prove that $(f(x_n))$ is Cauchy.

   (c) If $f : A \to \mathbb{R}$ is uniformly continuous and in addition, $A \subset \mathbb{R}$ is a bounded set, show that f is bounded.

5. (a) State the Mean Value Theorem.

   (b) Suppose $f : \mathbb{R} \to \mathbb{R}$ has a derivative at all points and that $f(c) = 0$ for some $c \in \mathbb{R}$. Show that $g(x) = |f(x)|$ has a derivative at c if and only if $f'(c) = 0$.

   (c) Let $f : [0, 1] \to \mathbb{R}$ be continuous on $[0, 1]$ and differentiable on $(0, 1)$. If $f(0) = 0$ and f' is decreasing on $(0, 1)$, prove that

      i. $\dfrac{f(x)}{x} \geq f'(x)$ for $0 < x < 1$ and

      ii. the function $g : (0, 1) \to \mathbb{R}$ defined by $g(x) = \dfrac{f(x)}{x}$ is decreasing on $(0, 1)$.

6. (a) State the Bolzano-Weierstrasse Theorem.

   (b) Suppose $A \subset \mathbb{R}$, prove that either A is bounded below or there exists a sequence (x_n), $x_n \in A$, such that $\lim x_n = -\infty$.

   (c) Let $f : [a, b] \to \mathbb{R}$ be continuous. Assume that such a function is bounded. Prove that f attains its sup and inf in $[a, b]$.

7. (a) State the Location of Roots Theorem.

   (b) Suppose $p(x) = a_{2m}x^{2m} + a_{2m-1}x^{2m-1} + \cdots + a_1x + a_0$ for some $m \in \mathbb{N}$ (i.e. a polynomial function of even degree) such that $(a_{2m})(a_0) < 0$. Show that there are at least two values $t_1, t_2 \in \mathbb{R}$ such that $p(t_1) = 0 = p(t_2)$.
   (Hint First consider the case when $a_{2m} > 0$.)

McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 189-242A

ANALYSIS I

Examiner: Professor K.N. GowriSankaran
Associate Examiner: Professor S.W. Drury

Date: Tuesday, December 15, 1998
Time: 2:00 P.M. - 5:00 P.M.

INSTRUCTIONS

CALCULATORS ARE NOT PERMITTED.
All questions count equally.

This exam comprises the cover and 2 pages of questions.