1. (a) Define the concepts '(a_n) converges to a' and '(b_n) tends to ∞'.
   (b) Let $0 < a \leq b \leq c$. Prove that $([a^n + b^n + c^n]^{1/n})$ converges to c.
   (c) Suppose $x_n \geq 0$ for all n and that $((-1)^n x_n)$ converges. Prove that (x_n) converges.

2. (a) Let (a_n) be a monotone increasing sequence. Prove that either (a_n) converges to $a \in \mathbb{R}$ or (a_n) tends to ∞.
   (b) Let (a_n) be an increasing sequence which converges to a. Define

$$b_n := \frac{1}{n}[a_1 + a_2 + \cdots + a_n]$$

   for all $n \in N$. Prove that (b_n) is increasing and converges to a.

3. (a) State the Mean Value Theorem.
   (b) If f is differentiable on an interval I and f' is bounded, prove that f satisfies a Lipschitz condition and hence that f is also uniformly continuous on I.
   (c) Hence or otherwise prove that $x \mapsto \sin x$ is uniformly continuous on $\mathbb{R}$.
   (d) Let f be a monotone function on $[a, b]$ such that f satisfies the intermediate value property. Show that f is continuous.

4. A differentiable function $f : I \to R$ (where $I := [a, b]$) is said to be uniformly differentiable if for every $\varepsilon > 0$ there exists a $\delta > 0$, such that $\left| \frac{f(x) - f(y)}{x - y} - f'(x) \right| < \varepsilon$ whenever $|x - y| < \delta$ and $x, y \in I$.
   Prove that f is uniformly differentiable if and only if f' is continuous on I.

5. (a) Suppose f is uniformly continuous on $[a, b]$ and on $[b, c]$. Prove that f is uniformly continuous on $[a, c]$.
   (b) Let $f : A \to \mathbb{R}$ be uniformly continuous and further for every $x \in A$, $|f(x)| \geq k > 0$. Prove that $\frac{1}{f}$ is uniformly continuous on A.
   (c) Give an example of a uniformly continuous function f such that $\frac{1}{f}$ is not uniformly continuous.

6. (a) Let I be an interval and $f : I \to \mathbb{R}$ continuous. Further, let $|f(x)| > 0$ for all $x \in I$. Prove that either $f(x) > 0$ for all $x \in I$ or $f(x) < 0$ for all $x \in I$.
   (b) Let $p(x) := x^4 + ax^3 + bx^2 + cx + 1$. Show that $\lim_{x \to \pm\infty} p(x) = \infty$. Hence, prove that there exists a $x_0 \in \mathbb{R}$ such that $p(x_0) = \text{Inf.}\ \{p(x) : x \in \mathbb{R}\}$.
   [Hint: Find a L such that $|x| \geq L \Rightarrow p(x) \geq 2$ and note that $p(0) = 1$].

<u>FACULTY OF SCIENCE</u>

<u>FINAL EXAMINATION</u>

<u>MATHEMATICS 189-242A</u>

<u>ANALYSIS I</u>

Examiner: Professor K. GowriSankaran Date: Thursday, December 11, 1997
Associate Examiner: Professor S. Drury Time: 2:00 P.M. - 5:00 P.M.

<u>INSTRUCTIONS</u>

All questions count equally though the level of difficulty may vary.
NO CALCULATORS PERMITTED

This exam comprises the cover and 1 page of questions.