NOTE TO PRINTER

(These instructions are for the printer. They should not be duplicated.)

THIS EXAMINATION SHOULD BE PRINTED ON $8\frac{1}{2} \times 14$ PAPER, AND STAPLED WITH 3 SIDE STAPLES, SO THAT IT OPENS LIKE A LONG BOOK.

# McGILL UNIVERSITY
# FACULTY OF SCIENCE
# FINAL EXAMINATION

## MATHEMATICS 189–240A

## DISCRETE STRUCTURES AND COMPUTING

EXAMINER: Professor W. G. Brown DATE: December 9th, 1999
ASSOCIATE EXAMINER: Professor W. O. J. Moser TIME: 09:00 – 12:00 hours

FAMILY NAME:
MR, MISS, MS, MRS, &c.:
GIVEN NAMES:

STUDENT NUMBER: COURSE AND YEAR:

## Instructions

1. Fill in the above clearly.
2. Do not tear pages from this book; all your writing — even rough work — must be handed in.
3. Calculators are not permitted.
4. This examination booklet consists of this cover, Pages 1 through 9 containing questions; and Pages 10 and 11, which are blank.
5. Show all your work. All solutions are to be written in the space provided on the page where the question is printed. When that space is exhausted, you may write *on the facing page*. Any solution may be continued on the last pages, or the back cover of the booklet, but you must indicate any continuation clearly on the page where the question is printed!
6. You are advised to spend the first few minutes scanning the problems. (Please inform the invigilator if you find that your booklet is defective.)

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|---|---|---|---|---|---|---|---|
| 1 /10 | 2(a) /2 | 2(b) /3 | 2(c) /6 | 2(d) /3 | 3 /4 | 4 /10 | 5 /12 |
| 6 /12 | 7 /12 | 8 /12 | 9(a) /2 | 9(b) /6 | 9(c) /6 | | |
| | | | | Raw | Scaled | TERM | |

1. [10 MARKS] Using any method studied in this course, and showing all of your work, determine whether or not the following is a valid rule of inference:

$$\begin{array}{c} \neg p \to q \\ \neg q \\ p \to (r \wedge s) \\ \hline \therefore r \vee s \end{array}$$

2. (a) [2 MARKS] Define what is meant by an *equivalence relation* on a set.

   (b) [3 MARKS] Showing all your work, determine, for any positive integer $n$, exactly how many (binary) relations there are on the set $\{1, 2, ..., n\}$.

   (c) [6 MARKS] Showing all your work, determine exactly how many equivalence relations there are on the 4-element set $S = \{a, b, c, d\}$.

   (d) [3 MARKS] Showing all your work, determine exactly how many total orderings there are on the 4-element set $S$ defined above.

3. [4 MARKS] Prove or disprove: *Among any 101 integers in the set* $\{n | 101 \leq n \leq 300\}$, *there must exist two integers* $a$ *and* $b$ *such that* $a|b$.

4. [10 MARKS] Prove by induction, or disprove by providing a counterexample: For all integers $N \geq 2$,

$$\sum_{n=2}^{N} \left( n^2 - n - \frac{1}{n^2 - n} \right) = \frac{N^3 - N}{3} - \frac{1}{N}$$

5. [12 MARKS] Your are to solve this problem *only by using exponential generating functions* – no other method will be accepted. You have a supply of 3 kinds of letters from which $n$-letter words are to be formed, with no restrictions: you may use each of the 3 letters any non-negative number of times in any word. Determine the exponential generating function for the number, $a_n$, of words that can be formed; and use this to determine $a_n$ in "closed form" (without using any summation signs).

6. [12 MARKS] Showing all your work, determine, for any integer $n$, a formula for the number of solutions $(x_1, x_2, x_3, x_4)$ to the inequality

$$x_1 + x_2 + x_3 + x_4 \leq n$$

in non-negative integers which satisfy all of the following constraints simultaneously:

- $3 \leq x_1 \leq 6$
- $x_2 > 2$
- $x_3 > 4$
- $0 \leq x_4 \leq 3$

Verify that your formula is correct for $n \leq 5$. (It is not necessary to simplify the formula.)

7. [12 MARKS] Showing all your work, determine the value of $a_n$, the general term in a sequence $\{a_n\}$ which satisfies the recurrence

$$2a_{n+2} - 3a_{n+1} + a_n = 2^{1-n} \qquad (n \geq 0) \tag{1}$$

subject to the initial conditions

$$a_0 = 0 \tag{2}$$

$$a_1 = 0 \tag{3}$$

8. [12 MARKS] The Petersen graph has vertex-set

$$V = \{a, b, c, d, e, v, w, x, y, z\},$$

and edge-set

$$E = \{ab, bc, cd, de, ea, av, bw, cx, dy, ez, vx, xz, zw, wy, yv\}.$$

Use the Euler polyhedron formula (not the Kuratowski Theorem) to prove that the Petersen graph is not planar. (You may assume that it is known that the Peterson graph has no circuits of lengths 3 or 4.)

9. (a) [2 MARKS] For finite trees on more than 1 vertex the number of vertices of degree 1 is bounded below. State the best bound.

(b) [6 MARKS] Proceeding systematically, develop a list of trees on $n$ vertices, where $n = 1, 2, ..., 6$.

(c) [6 MARKS] For each of the trees listed in (b), determine the number of ways of labelling the vertices with distinct labels 1, 2, ..., $n$. Explain your reasoning in every case.

CONTINUATION PAGE FOR PROBLEM NUMBER

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