1. (6%) Identify which of the following are true for all sets A and B. Justify your answers by giving a proof in case it is always true, and a counterexample otherwise.

   (a) $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$.

   (b) $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.

   (c) $A \times (B \cup A) = (A \times B) \cup (A \times A)$.

2. (8%) Suppose that m and n are natural numbers with $m < n$.

   Show that $\binom{2n}{m} < \binom{2n}{n}$.

3. (7%) Suppose that G is a commutative group and H and K are subgroups of G.

   (a) Show that the set $HK = \{hk \in G : h \in H, k \in K\}$ is also a subgroup of G.

   (b) Give a counterexample in case G is not assumed to be commutative.

   [Hint: You might want to consider the matrices $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.]

4. (9%) Which of the following is a subring of $\mathcal{Z}[x]$? Which is an ideal? Justify your answers.

   (a) The set of all polynomials in $\mathcal{Z}[x]$ of degree at least 3, together with 0.

   (b) The set $\{a_3x^3 + a_4x^4 + \cdots + a_kx^k : a_3, a_4, \cdots, a_k \in \mathcal{Z}\}$.

   (c) The set $\{a_0 + a_2x^2 + a_4x^4 + \cdots + a_{2n}x^{2n} : a_0, a_2, \cdots, a_{2n} \in \mathcal{Z}\}$.

5. (7%) Show that $\{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathcal{Z}_2\}$ is a ring with unity. How many elements does it have? Is it commutative? What are the units (i.e., the elements with multiplicative inverses in the ring)? Justify everything.

6. (7%) Suppose that R and S are rings, I is an ideal of R and J is an ideal of S. Show that $I \times J$ is an ideal of $R \times S$ and that $(R \times S)/(I \times J) \cong (R/I) \times (S/J)$.

7. (8%) Find a subfield F of $\mathcal{C}$ isomorphic to $\mathcal{Q}/(x^2 + 13)$; give two different isomorphisms between F and $\mathcal{Q}/(x^2 + 13)$.

8. (10%) Suppose that F is a field and that the nonconstant polynomials $P(x)$ and $Q(x)$ in $F[x]$ are relatively prime. Suppose that $R(x)$ and $S(x)$ are any polynomials in $F[x]$. Prove that there is a polynomial $T(x) \in F[x]$ such that

$$T(x) \equiv R(x)(mod\ P(x)) \text{ and } T(x) \equiv S(x)(mod\ Q(x)).$$

9. (9%) (a) Find the minimal polynomial for the complex number $\sqrt{2} + 2i\sqrt{2}$

   (i) over $\mathcal{Q}$ and then

   (ii) over $\mathcal{R}$. In each case explain why the polynomial is irreducible.

   (b) Which subfield of $\mathcal{C}$ is a splitting field over $\mathcal{Q}$ for this polynomial?

10. (12%) Suppose that $h : \mathcal{Z}[x] \longrightarrow \mathcal{C}$ is the homomorphism such that $h(n) = n$ for all $n \in \mathcal{Z}$ and $h(x) = -2i$.

    (a) Give an explicit formula for $h(P)$, where P is any polynomial over $\mathcal{Z}$.

    (b) What is the range S of h?

    (c) What is $K = ker(h)$?

    (d) Give an isomorphism from $\mathcal{Z}[x]/K$ onto S.

11. (6%) List all the maximal ideals of $\mathcal{Z}_{20}$. For each such ideal I, give a natural number n so that $\mathcal{Z}_{20}/I$ is isomorphic to $\mathcal{Z}_n$.

12. (10%) Find all the rational roots of

$$2x^6 - 13x^5 + 26x^4 - 80x^3 + 145x^2 - 105x + 25.$$

McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 189-235A

BASIC ALGEBRA I

Examiner: Professor J. Loveys | Date: Monday, December 14, 1998
Associate Examiner: Professor H. Darmon | Time: 9:00 A.M. - 12:00 Noon.

INSTRUCTIONS

Calculators are not permitted.

In this exam, you may use any results proved in class, on the assignments or on the midterm. Include all your work – anything you want considered for marks – on the booklet(s) provided.

Good luck!

This exam comprises the cover and 2 pages of questions.