Final Examination December 21, 1995 Mathematics 189-223A

PART I. Group 1

1. (a) Give the general solution of the following system of differential equations:

$$\begin{aligned} \frac{dx}{dt} &= 4x + 2y \\ \frac{dy}{dt} &= -3x - y \end{aligned}$$

(b) Given that $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are eigenvectors of the linear transformation $T : V \to V$ for the eigenvalues λ_1, λ_2, λ_3. Prove that if the eigenvalues are pairwise distinct, then the three eigenvectors form a linearly independent set.

2. Let $U = \text{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ be the subspace of $\mathbf{R}^4$ spanned by the vectors

$$\mathbf{v}_1 = (1,0,1,0)^t, \mathbf{v}_2 = (-1,1,0,0)^t, \mathbf{v}_3 = (0,-2,1,1)^t$$

(a) Give a basis of U which is orthogonal with respect to the ordinary inner product (dot–product).

(b) Let $P : \mathbf{R}^4 \to \mathbf{R}^4$ be the operator projecting every vector in $\mathbf{R}^4$ perpendicularly (i.e. orthogonally) onto U.

Compute the third column of the matrix of P relative to the standard basis of $\mathbf{R}^4$. (Do not calculate the entire matrix.)

(c) Give an orthonormal basis $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \mathbf{f}_4\}$ of $\mathbf{R}^4$ such that $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3 \in U$.

3. In the standard classification, decide what kind of quadric surface is given by the equation

$$x^2 + y^2 + z^2 - 2xy - 2xz - 4yz - 2x + 14y - 2z = 0 \ .$$

If applicable, determine the (x, y, z) co-ordinates of the center of this quadric.

4. Let $U = \mathrm{Span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, $\quad V = \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ where

$$\mathbf{u}_1 = (1, 2, -1, 0)^t,\ \mathbf{u}_2 = (0, 1, 2, 1)^t,\ \mathbf{u}_3 = (-1, 0, 5, 2)^t \quad \text{and}$$

$$\mathbf{v}_1 = (3, -1, 2, 0)^t,\ \mathbf{v}_2 = (4, 2, 3, 1)^t.$$

Determine bases and dimensions for $U,\ V,\ U \cap V,\ U + V$.

PART II. Group 1

Questions on this part of the examination are to be answered on the multiple choice score sheet provided.

1. The non–degenerate conic section is given in Cartesian co–ordinates by the equation

$$2x^2 + 2\sqrt{2}xy + 3y^2 + x - y = 0$$

The associated symmetric matrix has one of its eigenvalues equal to 1. Which of the following statements is true about this curve?

(a) a hyperbola with principal axes parallel to the lines $\sqrt{2}x - y = 0, \qquad x + \sqrt{2}y = 0.$

(b) a parabola with principal axes parallel to the lines $y - 2\sqrt{2}x = 0, \qquad 2\sqrt{2}y + x = 0.$

(c) an ellipse with principal axes parallel to the lines $y - \sqrt{2}x = 0, \qquad \sqrt{2}y + x = 0$

(d) an ellipse with principal axes parallel to the lines $y - x = 0, \qquad y + x = 0.$

(e) a hyperbola with principal axes parallel to the lines $y - x = 0, \qquad y + x = 0.$

2. The necessary and sufficient condition on the parameters a, b for the system

$$\begin{array}{rcrcrcl} ax & - & y & + & bz & = & 1 \\ & & 2y & + & az & = & 0 \\ 2ax & - & y & & & = & 0 \end{array}$$

to have a unique solution is:

(a) $a^2b + b^2 = 0.$

(b) $-4ab - a^2 \neq 0.$

(c) $ab + 2 \neq 0.$

(d) $-a^2b + 2b^2 \neq 0.$

(e) $a^2 - 2ab = 0.$

3. Let $A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$, and $B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

(a) A is not diagonalisable, but B is diagonalisable over **R**.

(b) Neither A nor B is diagonalisable over **R**.

(c) Both A and B are diagonalisable over **R**.

(d) A is diagonalisable and B is not diagonalisable over **R**.

(e) Both A and B are diagonalisable over **C**.

5. U is a vector space, V and W are subspaces of U. Exactly one of the states of affairs listed below regarding the dimensions of U, V, W, $V \cap W$, $V + W$ is possible. Which one is it?

(a) $\dim U = 6$, $\dim V = 4$, $\dim W = 4$, $\dim V \cap W = 1$, $\dim V + W = 7$.

(b) $\dim U = 6$, $\dim V = 5$, $\dim W = 3$, $\dim V \cap W = 4$, $\dim V + W = 4$.

(c) $\dim U = 6$, $\dim V = 3$, $\dim W = 2$, $\dim V \cap W = 1$, $\dim V + W = 3$.

(d) $\dim U = 7$, $\dim V = 5$, $\dim W = 4$, $\dim V \cap W = 2$, $\dim V + W = 7$.

(e) $\dim U = 7$, $\dim V = 4$, $\dim W = 3$, $\dim V \cap W = 0$, $\dim V + W = 6$.

6. Given the linearly independent vectors $\mathbf{v}$ and $\mathbf{w}$, which of the following statements below is always correct (here $< \, , \, >$ denotes the usual inner product on $\mathbf{R}^n$) for the vector $\mathbf{u}$ defined by

$$\mathbf{u} = <\mathbf{w}, \mathbf{w}> \mathbf{v} - <\mathbf{v}, \mathbf{w}> \mathbf{w}?$$

(a) $\mathbf{v}$ is orthogonal to $\mathbf{w}$.

(b) $\mathbf{u}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$.

(c) $\mathbf{u}$ is a unit vector.

(d) $\mathbf{v}$ is orthogonal to $\mathbf{u}$.

(e) $\mathbf{u}$ is orthogonal to $\mathbf{w}$.

7. Given the basis $B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} \right\}$ of $\mathbf{R}^4$. the co-ordinate vector of $\mathbf{v} = (6, -3, -2, 3)^t$ relative to this basis is

(a) $\begin{bmatrix} 1/2 \\ 3/4 \\ -3/2 \\ 1/4 \end{bmatrix}$ (b) $\begin{bmatrix} 6 \\ -3 \\ -2 \\ -1 \end{bmatrix}$ (c) $\begin{bmatrix} 2 \\ 2 \\ -3 \\ 2 \end{bmatrix}$ (d) $\begin{bmatrix} -1/3 \\ 5/4 \\ 3/\sqrt{3} \\ 0 \end{bmatrix}$ (e) $\begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix}$.

8. Which of the following is a two dimensional subspace of $\mathbf{R}^4$?

(a) $\{(x, y, z, w)^t | x + 2y - 3z + w = 2\}$.

(b) $\{(x, y, z, w)^t | x^2 - y^2 + 3w^2 = 0\}$.

(c) $\{(x, y, z, w)^t | x + y - z + w = 0\}$.

(d) the column space of C where $C = \begin{pmatrix} 1 & 3 & -1 & 4 \\ -1 & 3 & -5 & 2 \\ 3 & 4 & 2 & 7 \\ 1 & 3 & -1 & 4 \end{pmatrix}$

(e) The solution space of A where $A = \begin{pmatrix} 1 & -1 & 3 & 6 \\ 4 & -2 & 2 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$