1. (a) Find the slope of the tangent to the curve with equation $x^3 - 3xy^2 + y^3 = 1$ at the point $(2, -1)$.

   (b) Find the values of $A$ and $B$, given that the function $f(x) = \dfrac{A}{\sqrt{x}} + B\sqrt{x}$ has a minimum value of 6 on $(0, \infty)$ at the point $x = 9$.

   (c) Evaluate the following limits

   (i) $\displaystyle\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right)$, (ii) $\displaystyle\lim_{x\to 0^+}\sqrt{x}\sec x$, (iii) $\displaystyle\lim_{x\to\infty}[(e^x + x)^{1/x}]$.

2. (a) The point $(2, 1, 2)$ is common to the two surfaces with equations $x^2y^2 + 2x + z^3 = 16$ and $3x^2 + y^2 - 2z = 9$. Find a parametric equation of the line of intersection of the tangent planes at $(2, 1, 2)$ to the two surfaces.

   (b) Find the direction in which $f(x, y, z) = ze^{xy}$ increases most rapidly at the point $(0, 1, 2)$. What is the maximum rate of increase?

   (c) If $z = f(x, y)$ and $x = r\cos\theta,\ y = r\sin\theta$ prove (using chain rule) that

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2.$$

3. (a) For each of the functions below (i) find all the critical points, and (ii) classify the nature (local max/min) of the critical points. Indicate where the function increases/decreases, specify the points of inflection, and determine any vertical or horizontal asymptotes. Finally, sketch a graph of the function, indicating the concavity.

   (i) $f(x) = \dfrac{1}{4}\left[x^3 - \dfrac{3x^2}{2} - 6x + 2\right]$.

   (ii) $f(x) = \dfrac{3 - x^2}{1 - x^2}$.

   (b) A falling stone is, at a certain instant, 100ft above the ground. Two seconds later it is only 16ft above the ground. If the stone was thrown down with an initial speed 5ft/sec, from what height was it thrown?

4. (a) A power line is needed to connect a power station on the shore of a river to an island 4km downstream and 1km offshore. Find the minimum cost for such a line given that it costs \$50,000/km to lay wire under water and \$30,000/km to lay wire under ground.

   (b) A man standing 3ft from the base of a lamp post casts a shadow 4ft long. If the man is 6ft tall and walks away from the lamp at a speed of 400ft/min, at what rate will his shadow lengthen?

   (c) An accident in a nuclear plant has left the surrounding area polluted with a radioactive element that decays at a rate proportional to its current amount $A(t)$. The initial radiation level is $10S$, where $S$ is the maximum level of radiation that is safe. 100 days later the radiation level is $7S$. How long after the accident will it be before it is safe for people to return to the area?

5. (a) Find the critical points of $x^2 + \frac{2}{3}xy + y^2 + \frac{576}{x} + \frac{576}{y}$ in the region $x > 0,\ y > 0$. Use the second derivative test to decide if there is local max/min at each of the critical point(s).

   (b) Use the Lagrange multiplier method to find the maximum volume of a closed rectangular box, if the sum of the areas of its six sides equals 600 sq.cm.

McGILL UNIVERSITY

FACULTY OF SCIENCE

# FINAL EXAMINATION

## MATHEMATICS 189-150A

## CALCULUS A

Examiner: Professor K.N. GowriSankaran
Associate Examiner: Professor W.G. Brown

Date: Friday, December 17, 1999
Time: 2:00 P.M. - 5:00 P.M.

INSTRUCTIONS

**Calculators are not permitted.**
**All questions count equally though the level of difficulty may vary.**
**Please write down the name of the instructor of your tutorial section.**

This exam comprises the cover and two pages of questions.